6.1 Equivalent Rational Expressions

Lesson Organizer
60 – 75 min

Key Math Concepts
An equivalent form of a rational expression can be written by multiplying or dividing the numerator and denominator by the same monomial or binomial. The values of the variable for which the expressions are undefined must be identified.

Curricular Competencies: RM, US, CR1, CR2

Student Materials
- Master 6.1a (*optional*)

Vocabulary
rational expression, non-permissible values, equivalent rational expressions

FOCUS Determine equivalent forms of rational expressions.

Get Started

For which values of x is each expression defined?

$\frac{12}{x}$	$10 \cdot x$	$3 + x$	$5 \div x$
$x \neq 0, x \in \mathbb{R}$	$x \in \mathbb{R}$	$x \in \mathbb{R}$	$x \neq 0, x \in \mathbb{R}$

Construct Understanding US, CR1

Evaluate each expression, where possible, for $x = 2$, $x = 0$, and $x = 1$.
What observations can you make about the allowable values of the variables in these expressions?

Expression	$x = 2$	$x = 0$	$x = 1$
$\frac{6}{x}$	$\frac{6}{2} = 3$	$\frac{6}{0}$; undefined	$\frac{6}{1} = 6$
$\frac{4x}{x + 2}$	$\frac{4(2)}{2 + 2} = 2$	$\frac{4(0)}{0 + 2} = 0$	$\frac{4(1)}{1 + 2} = \frac{4}{3}$
$\frac{3}{x - 2}$	$\frac{3}{0}$; undefined	$\frac{3}{0 - 2} = -\frac{3}{2}$	$\frac{3}{1 - 2} = -3$
$\frac{x}{x^2 - 6x + 8}$	$\frac{2}{2^2 - 6(2) + 8} = \frac{2}{0}$; undefined	$\frac{0}{0^2 - 6(0) + 8} = 0$	$\frac{1}{1^2 - 6(1) + 8} = \frac{1}{3}$
$\frac{3}{(x - 1)(x + 2)}$	$\frac{3}{(2 - 1)(2 + 2)} = \frac{3}{4}$	$\frac{3}{(0 - 1)(0 + 2)} = -\frac{3}{2}$	$\frac{3}{(1 - 1)(1 + 2)} = \frac{3}{0}$; undefined

In each expression, x cannot have a value that makes the denominator 0 because I cannot divide by 0; the expression is undefined. For example, when I substituted $x = 2$ in $\frac{3}{x-2}$, the denominator became 0. It is not possible to evaluate the expression when the value of the denominator is 0.

When the numerator and denominator of a fraction are integers, the fraction is a rational number.
When the numerator and denominator of a fraction are polynomials, the fraction is a **rational expression**.

A **rational expression** is an algebraic expression that can be written as the quotient of two polynomials.

These are rational expressions:

$\frac{2x + 3}{5x + 4}$ $\frac{x^2 - 6}{3xy}$ $\frac{x^2 - 9}{x^2 + 8x + 15}$

A rational expression cannot contain roots of variables, or variables as exponents. These expressions are *not* rational expressions:

$\frac{2^x + 3}{4x - 2}$ $\frac{x^2 + 4}{2\sqrt{x}}$

Rational expressions are not defined for values of the variable that make the denominator 0. These values are called **non-permissible values**.
The expression $\frac{x + 5}{x - 3}$ is not defined for $x = 3$. So, $x = 3$ is a non-permissible value of $\frac{x + 5}{x - 3}$.

CR1, CR2

THINK FURTHER

Why is every polynomial expression a rational expression?

A polynomial expression can be written as a rational expression with denominator 1.

Example 1 Determining Non-Permissible Values

Determine the non-permissible values for each rational expression.

a) $\frac{x^2 + 2}{x^2 - x - 6}$ **b)** $\frac{x}{x^2 + 1}$

SOLUTION

a) Equate the denominator to 0, then solve the equation.

$x^2 - x - 6 = 0$ Factor.

$(x - 3)(x + 2) = 0$

So, $x - 3 = 0$ or $x + 2 = 0$

$x = 3$ $x = -2$

The non-permissible values of $\frac{x^2 + 2}{x^2 - x - 6}$ are $x = 3$ and $x = -2$.

b) Since the square of a number is always positive, $x^2 \geq 0$.

So, $x^2 + 1 > 0$

Since the denominator cannot equal 0, there are no non-permissible values.

Check Your Understanding

1. Determine the non-permissible values for each rational expression.

a) $\frac{5x}{x^2 - 9}$ b) $\frac{3x + 2}{x^2 - 8x + 16}$

Equate each denominator to 0.

a) $x^2 - 9 = 0$

$x^2 = 9$

So, $x = 3$ and $x = -3$ are the non-permissible values.

b) $x^2 - 8x + 16 = 0$

$(x - 4)(x - 4) = 0$

$x - 4 = 0$

$x = 4$

So, $x = 4$ is the non-permissible value.

Check Your Understanding

Answers:

1. a) $x = 3$ and $x = -3$

b) $x = 4$

THINK FURTHER CR1

Is the expression $\frac{2x}{2x^3 + 2}$ defined for all values of x? Explain.

No, since the cube of a negative number is negative, it is possible for the denominator to equal 0: $2x^3 + 2 = 0$ when $x = -1$. The expression $\frac{2x}{2x^3 + 2}$ has a non-permissible value of $x = -1$.

TEACHER NOTE

When the denominator of a rational expression is in factored form, students can use mental math to identify the non-permissible values.

To write an equivalent form of a rational number, multiply or divide the numerator and denominator by the same number. For example,

$$\frac{12}{18} = \frac{12 \times 2}{18 \times 2} = \frac{24}{36} \qquad \frac{12}{18} = \frac{12 \div 3}{18 \div 3} = \frac{4}{6}$$

$\frac{12}{18}$, $\frac{24}{36}$, and $\frac{4}{6}$ are equivalent rational numbers.

The same strategy is used to write an equivalent form of a rational expression: multiply or divide the numerator and denominator by the same monomial or binomial.

For example,

$$\frac{20xy}{8x^2} = \frac{20xy \cdot x}{8x^2 \cdot x} = \frac{20x^2y}{8x^3} \qquad \frac{20xy}{8x^2} = \frac{20xy \div 2x}{8x^2 \div 2x} = \frac{10y}{4x}$$

Each expression has $x = 0$ as a non-permissible value.

So, $\frac{20xy}{8x^2}$, $\frac{20x^2y}{8x^3}$, and $\frac{10y}{4x}$ are **equivalent rational expressions** for $x \neq 0$.

When stating that rational expressions are equivalent, the values of the variable for which the expressions are undefined must be identified.

Example 2 Writing Equivalent Forms of a Rational Expression

RM, US

Use multiplication and division to write two equivalent forms of the rational expression $\frac{3(x + 2)}{(x + 2)(x - 4)}$.

SOLUTION

The expression has $x = -2$ and $x = 4$ as non-permissible values.

- Multiply the numerator and denominator by the same monomial or binomial. One possible choice is $x - 5$.

$$\frac{3(x + 2)}{(x + 2)(x - 4)} = \frac{3(x + 2)}{(x + 2)(x - 4)} \cdot \frac{(x - 5)}{(x - 5)} = \frac{3(x + 2)(x - 5)}{(x + 2)(x - 4)(x - 5)}$$

This rational expression has an additional non-permissible value: $x = 5$

Check Your Understanding

2. Use multiplication and division to write two equivalent forms of the rational expression $\frac{(x + 5)(x - 1)}{2(x - 1)}$.

The expression has $x = 1$ as a non-permissible value.

- $$\frac{(x + 5)(x - 1)}{2(x - 1)} = \frac{(x + 5)(x - 1)}{2(x - 1)} \cdot \frac{(x + 5)}{(x + 5)} = \frac{(x + 5)^2(x - 1)}{2(x - 1)(x + 5)}$$

▶

Check Your Understanding

Answer:

2. sample answers: $\frac{(x + 5)^2(x - 1)}{2(x - 1)(x + 5)}$, $x \neq 1, -5$; $\frac{x + 5}{2}$, $x \neq 1$

- Divide the numerator and denominator by the same monomial or binomial. One possible choice is the common factor $x + 2$.

$$\frac{3(x+2)}{(x+2)(x-4)} = \frac{3\cancel{(x+2)}}{\cancel{(x+2)}(x-4)}$$ Include the non-permissible values of both expressions.

$$= \frac{3}{x-4},\ x \neq -2, 4$$

The equivalent expressions are: $\frac{3(x+2)(x-5)}{(x+2)(x-4)(x-5)}$; $x \neq -2, 4, 5$ and $\frac{3}{x-4}$, $x \neq -2, 4$

There is an additional non-permissible value: $x = -5$

- $$\frac{(x+5)(x-1)}{2(x-1)} = \frac{(x+5)\cancel{(x-1)}}{2\cancel{(x-1)}}$$

$$= \frac{x+5}{2},\ x \neq 1$$

The equivalent expressions are: $\frac{(x+5)^2(x-1)}{2(x-1)(x+5)}$, $x \neq 1, -5$; and $\frac{x+5}{2}$, $x \neq 1$

In *Example* 2, the numerator and denominator of a rational expression are divided by a common factor. All non-permissible values from any step of the solution must be stated.

A rational number is in simplest form when the numerator and denominator have no common factors other than 1. The same is true for rational expressions. It may be necessary to write the numerator and denominator in factored form to identify common factors.
When two rational expressions are equal, they are equal for all the values of the variable for which both expressions are defined.

Example 3 Simplifying Rational Expressions

RM, US Animation

Write each rational expression in simplest form.

a) $\frac{15x^2yz^3}{20xyz}$ **b)** $\frac{3x^2 - 8x}{2x}$

c) $\frac{x^2 + 6x + 8}{x^2 - 4}$ **d)** $\frac{16 - x^2}{2x^2 - 11x + 12}$

SOLUTION

For each expression:
Factor the numerator and denominator.
Identify the non-permissible values of the variables.
Divide the numerator and denominator by their common factors.

a) The non-permissible values are $x = 0$, $y = 0$, and $z = 0$.

$$\frac{15x^2yz^3}{20xyz} = \frac{\cancel{5xyz}(3xz^2)}{\cancel{5xyz}(4)}$$

$$= \frac{3xz^2}{4}$$

So, $\frac{15x^2yz^3}{20xyz} = \frac{3xz^2}{4}$, $x \neq 0$, $y \neq 0$, $z \neq 0$

Check Your Understanding

3. Write each rational expression in simplest form.

a) $\frac{-25a^3b^2c}{35ab^5}$

b) $\frac{6x^2 + 12x}{3x}$

c) $\frac{x^2 - 49}{x^2 - 5x - 14}$

d) $\frac{2x^2 - 5x - 3}{9 - x^2}$

Check Your Understanding

Answers:

3. a) $\frac{-5a^2c}{7b^3}$, $a \neq 0$, $b \neq 0$

b) $2(x + 2)$, $x \neq 0$

c) $\frac{x+7}{x+2}$, $x \neq -2, 7$

d) $-\frac{2x+1}{3+x}$, $x \neq -3, 3$

a) The non-permissible values are $a = 0$ and $b = 0$.

$$\frac{-25a^3b^2c}{35ab^5} = \frac{-\cancel{5ab^2}(5a^2c)}{\cancel{5ab^2}(7b^3)}$$
$$= \frac{-5a^2c}{7b^3}, a \neq 0, b \neq 0$$

b) The non-permissible value is $x = 0$.

$$\frac{6x^2 + 12x}{3x} = \frac{{}^2\cancel{6x}(x + 2)}{\cancel{3x}}$$
$$= 2(x + 2), x \neq 0$$

c) $$\frac{x^2 - 49}{x^2 - 5x - 14} = \frac{(x - 7)(x + 7)}{(x - 7)(x + 2)}$$

The non-permissible values are $x = 7$ and $x = -2$.

$$\frac{x^2 - 49}{x^2 - 5x - 14} = \frac{\cancel{(x - 7)}(x + 7)}{\cancel{(x - 7)}(x + 2)}$$
$$= \frac{x + 7}{x + 2}, x \neq -2, 7$$

d) $$\frac{2x^2 - 5x - 3}{9 - x^2} = \frac{(2x + 1)(x - 3)}{(3 - x)(3 + x)}$$

The non-permissible values are $x = 3$ and $x = -3$.

$$\frac{2x^2 - 5x - 3}{9 - x^2} = \frac{(2x + 1)(x - 3)}{(3 - x)(3 + x)}$$
$$= \frac{-1(2x + 1)\cancel{(3 - x)}}{\cancel{(3 - x)}(3 + x)}$$
$$= -\frac{2x + 1}{3 + x}, x \neq -3, 3$$

TEACHER NOTE

In *Example 3d*, remind students that $(a - x) = -1(x - a)$; this can be used to produce common factors.

b) The non-permissible value is $x = 0$.

$$\frac{3x^2 - 8x}{2x} = \frac{\cancel{x}(3x - 8)}{2\cancel{x}}$$
$$= \frac{3x - 8}{2}$$

So, $\frac{3x^2 - 8x}{2x} = \frac{3x - 8}{2}, x \neq 0$

c) $$\frac{x^2 + 6x + 8}{x^2 - 4} = \frac{(x + 2)(x + 4)}{(x - 2)(x + 2)}$$

The non-permissible values are $x = 2$ and $x = -2$.

$$\frac{x^2 + 6x + 8}{x^2 - 4} = \frac{\cancel{(x + 2)}(x + 4)}{(x - 2)\cancel{(x + 2)}}$$
$$= \frac{x + 4}{x - 2}$$

So, $\frac{x^2 + 6x + 8}{x^2 - 4} = \frac{x + 4}{x - 2}, x \neq -2, 2$

d) $$\frac{16 - x^2}{2x^2 - 11x + 12} = \frac{(4 - x)(4 + x)}{(2x - 3)(x - 4)}$$

The non-permissible values are $x = \frac{3}{2}$ and $x = 4$.

$$\frac{16 - x^2}{2x^2 - 11x + 12}$$
$$= \frac{(4 - x)(4 + x)}{(2x - 3)(x - 4)}$$
$$= \frac{-1\cancel{(x - 4)}(4 + x)}{(2x - 3)\cancel{(x - 4)}}$$
$$= \frac{-(4 + x)}{2x - 3}, \text{ or } -\frac{4 + x}{2x - 3}$$

Writing $(4 - x)$ as $-1(x - 4)$ produces a common factor of $(x - 4)$.

So, $\frac{16 - x^2}{2x^2 - 11x + 12} = -\frac{4 + x}{2x - 3}, x \neq \frac{3}{2}, 4$

Discuss the Ideas

CR1

1. Explain why $\frac{x + 7}{x + 1}$ cannot be simplified to $\frac{7}{1}$, or 7, but $\frac{7x}{x} = 7$, for $x \neq 0$.

I cannot divide by a term when the numerator and denominator are the sum or difference of terms. The expression $\frac{x + 7}{x + 1}$ cannot be simplified to 7 because x is not a factor of the numerator and of the denominator. The expression $\frac{7x}{x}$ can be simplified to 7 because x is a non-zero factor of both the numerator and the denominator. I can only divide by non-zero common factors, not common terms.

2. To write an equivalent form of a rational expression, why are both the numerator and denominator multiplied by the same non-zero factor?

When a rational expression is multiplied by 1, its value does not change. When I multiply the numerator and denominator by the same non-zero factor, I am multiplying the expression by 1.

3. Does a rational expression always have non-permissible values?

No, a rational expression whose denominator cannot equal 0 does not have any non-permissible values; for example, when the denominator is 3 or $x^4 + 5$.

Exercises

A

4. Which expressions are rational expressions? Justify your choices.

a) $\frac{x+3}{5}$ b) $\frac{\sqrt{x}+2}{4x}$ c) $\frac{a}{7}$

d) $\frac{3^x+9}{x^2-2}$ e) $\frac{3\sqrt[3]{b}+2b}{16-b^2}$ f) $\frac{x^2+2x-7}{x+3}$

Parts a, c, and f are rational expressions because each expression is the quotient of two polynomials. Parts b and e are not rational expressions because they each contain the root of a variable; part d is not a rational expression because it has a variable as an exponent.

TEACHER NOTE

Remind students that a rational expression cannot contain roots of variables, or variables as exponents.

TEACHER NOTE

Elaboration

Questions 5 and 6 address: identify non-permissible values.

5. Identify the non-permissible values of the variable for each rational expression.

a) $\frac{2-x^2}{x+5}$

$x + 5 = 0$
$x = -5$
So, $x = -5$ is the non-permissible value.

b) $\frac{x+1}{(x-2)(x+8)}$

$(x - 2)(x + 8) = 0$
$x - 2 = 0$ or $x + 8 = 0$
$x = 2$ or $x = -8$
So, $x = 2$ and $x = -8$ are the non-permissible values.

c) $\frac{x^2+5x-4}{-13(x-12)}$

$x - 12 = 0$
$x = 12$
So, $x = 12$ is the non-permissible value.

RM, CR1

6. Determine whether the given value of x is a non-permissible value for the rational expression. Explain how you know.

a) $\frac{3x+9}{(x-5)(x+6)}$; $x = 6$

No, $x = 6$ does not result in the denominator equal to 0.

b) $\frac{5x}{x^2-4}$; $x = -2$

Yes, $x = -2$ results in the denominator equal to 0.

c) $\frac{x+5}{x^2+3x+2}$; $x = -2$

Yes, $x = -2$ results in the denominator equal to 0.